Evaluate the sum

Let x, y be real and $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1$, evaluate the sum $x+y$.

Method 1

Let $u=x+\sqrt{x^2+1}$, then

$(u-x)^2=x^2+1 \Rightarrow u^2-2ux+x^2=x^2+1 \Rightarrow u^2-2ux=1 \Rightarrow x=\frac{u^2-1}{2u}$

Similarly let $v=y+\sqrt{y^2+1}$, $y=\frac{v^2-1}{2v}$

From the given $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1,$ $uv=1$, therefore $v=\frac{1}{u}$

$$x+y=\frac{u^2-1}{2u}+\frac{v^2-1}{2v}=\frac{u^2-1}{2u}+\frac{\left(\frac{1}{u}\right)^2-1}{2\left(\frac{1}{u}\right)}=\frac{u^2-1}{2u}+\frac{1-u^2}{2u}=0$$

Method 2

Note that $\frac{1}{x+\sqrt{x^2+1}}=-x+\sqrt{x^2+1},$

$\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1 \Longrightarrow y+\sqrt{y^2+1}=-x+\sqrt{x^2+1}$ ....(1)

$(1)^2$, $y^2+2y\sqrt{y^2+1}+y^2+1=x^2-2x\sqrt{x^2+1}+x^2+1$

$2y\left(y+\sqrt{y^2+1}\right)=2x\left(-x+\sqrt{x^2+1}\right)$

By (1), cancelling the none-zero factor in parenthesis, we have $2y=-2x$, $x+y=0.$

Method 3

Since $\frac{1}{x+\sqrt{x^2+1}}=-x+\sqrt{x^2+1},$ $\frac{1}{y+\sqrt{y^2+1}}=-y+\sqrt{y^2+1}$

From given, we have $y+\sqrt{y^2+1}=-x+\sqrt{x^2+1}$ and $x+\sqrt{x^2+1}=-y+\sqrt{y^2+1}.$
Adding together and cancelling the radicals, we have $x+y=0$.

Method 4

Since $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1$
Taking natural logarithm, $\ln\left(x+\sqrt{x^2+1}\right)+\ln\left(y+\sqrt{y^2+1}\right)=0$ ....(1)
Let $u=\ln\left(x+\sqrt{x^2+1}\right) \Longrightarrow e^u=x+\sqrt{x^2+1}$ ... (1)

$$e^{-u} = \frac{1}{x+\sqrt{x^2+1}} = -x + \sqrt{x^2+1} \quad \dots (2)$$

$\frac{(1)-(2)}{2}$, $\quad x = \frac{e^u - e^{-u}}{2} \quad \dots .(3)$

Similarly, let $v = \ln\left(y + \sqrt{y^2+1}\right)$, $y = \frac{e^v - e^{-v}}{2} \quad \dots .(4)$

By (1), $u + v = 0$, $v = -u$

Therefore $\quad x + y = \frac{e^u - e^{-u}}{2} + \frac{e^v - e^{-v}}{2} = \frac{e^u - e^{-u}}{2} + \frac{e^{-v} - e^{v}}{2} = 0$.

Advanced students may discern the connection of this problem with the ***hyperbolic function*** :

$\sinh x = \frac{e^x - e^{-x}}{2}$ and $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

$\sinh^{-1} x + \sinh^{-1} y = 0 \Longrightarrow x + y = 0$

Method 5

If we allow the use of ***complex numbers***, the following is quite a mysterious way.

Let $\quad x = i \sin\alpha$, $y = i \sin\beta$

It is even more mysterious that by choosing suitable complex α, β we can make x, y are real.

Since $\quad \left(x + \sqrt{x^2+1}\right)\left(y + \sqrt{y^2+1}\right) = 1$

Therefore $\quad \left(i \sin\alpha + \sqrt{-\sin^2\alpha + 1}\right)\left(i \sin\beta + \sqrt{-\sin^2\beta + 1}\right) = 1$

$(\cos\alpha + i \sin\alpha)(\cos\beta + i \sin\beta) = 1$

By de' Moivres theorem, $\quad \cos(\alpha + \beta) + i \sin(\alpha + \beta) = 1$

$\therefore \alpha + \beta = 2n\pi$, where n is an integer.

$\therefore\ x + y = i \sin\alpha + i \sin\beta = i\ \sin\alpha + i \sin(2n\pi - \alpha) = i\ \sin\alpha - i\ \sin\alpha = 0$

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